MATH 347: FUNDAMENTAL MATHEMATICS, FALL 2015

PRACTICE PROBLEMS FOR MITERM 3

1. Write the number $1043_{(5)}$ in 4-ary representation.

Solution. Although one can do this directly, we will first convert the 5-ary into our good ol' decimal representation and worry about converting that into 4-ary.

$1043_{(5)} = 1 \cdot 5^3 + 0 \cdot 5^2 + 4 \cdot 5^1 + 3 \cdot 5^0 = 125 + 0 + 20 + 3 = 148$. Now we find its 4-ary representation.

What's the largest power of 4 that fits into 148? It's $64 = 4^3$.

How many times does $64 = 4^3$ fit in 148? **2** times and the remainder is $148 - 2 \cdot 64 = 20$.

How many times does $4^2 = 16$ fit in 20? **1** time and the remainder is $20 - 1 \cdot 16 = 4$.

How many times does $4^1 = 4$ fit in 4? **1** time and the remainder is $4 - 1 \cdot 4 = 0$.

How many times does $4^0 = 1$ fit in 0? **0** times (funny how in English one says "one time" and "zero times").

Thus, the 4-ary representation of 148 is $2110_{(4)}$. Hence, we showed that $1043_{(5)} = 2110_{(4)}$. □

2. Let D be a set and let $f, g : D \to \mathbb{R}$ be bounded functions such that $\forall x \in D$, $f(x) \le g(x)$. For each of the following statements, either prove it or give a counter-example.

(a) $\sup f(D) \le \inf g(D)$.

Solution. This is false and here is a counter-example. Let $D := \{1, 3\}$, $f = \mathrm{id}_D$, i.e. $f(1) = 1$ and $f(3) = 3$, and let $g = f + 1$, i.e. $g(1) = 2$ and $g(3) = 4$. We certainly have that for every $x \in D$, $f(x) \le g(x)$, but $\sup f(D) = \sup \{1, 3\} = 3$, while $\inf g(D) = \inf \{2, 4\} = 2$. □

(b) $\sup f(D) \le \sup g(D)$.

Solution. This is true. Put $u := \sup g(D)$. Recall that the supremum of a set is its least upper bound; in particular, it is less than or equal to *any* upper bound. Thus, to show $\sup f(D) \le u$, we only have to show that u is an upper bound for the set $f(D)$. By the definition of an upper bound (hope the reader has reviewed it by now), we have to show that for all $y \in f(D)$ $y \le u$. To this end, fix an arbitrary $y \in f(D)$. Being in $f(D)$ means that there is $x \in D$ such that $y = f(x)$. But for this x, we have $f(x) \le g(x)$. Moreover, $g(x) \in g(D)$, so $g(x) \le \sup g(D) = u$. Thus,

$$y = f(x) \le g(x) \le u,$$

□

and this is what we had to show.

(c) $\inf f(D) \le \inf g(D)$.

Solution. Analogous to the previous part (with inequalities reversed). □

3. Prove that for any sets $A, B \subseteq \mathbb{R}$ that are bounded above, $\sup(A \cup B) = \max \{\sup A, \sup B\}$.

Solution. Put $u := \max \{\sup A, \sup B\}$ and realize that all we have to show is that u is the supremum of the set $A \cup B$. By definition, we have to show two things:

(i) u is an upper bound for $A \cup B$.

Proof. We have to show that $\forall x \in A \cup B \ x \le u$. Fix arbitrary $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $x \le \sup A \le u$. If $x \in B$, then $x \le \sup B \le u$. Thus, either way, $x \le u$. □

(ii) Any number $v < u$ is not an upper bound for $A \cup B$.

Proof. Fix $v < u$. Because u is the maximum of $\sup A$ and $\sup B$, it is equal to one or the other, and we consider those cases separately. Suppose $u = \sup A$, i.e. u is the least upper bound of A, and since $v < u$, v is not an upper bound for A, which means that there is $x \in A$ with $x > v$. In particular, this x is in $A \cup B$, so v is not an upper bound for $A \cup B$. The case of $u = \sup B$ is handled similarly (these are symmetric cases). □

□

Before continuing further, let's review the definition of limit.

Definition 1. Let $P(n)$ be a mathematical statement for every $n \in \mathbb{N}$. We say that **eventually** $P(n)$ holds if there is (an event) $N \in \mathbb{N}$ such that for every (moment) $n \ge N$, $P(n)$ holds.

Definition 2. We say that $L \in \mathbb{R}$ is a *limit* of a sequence $(x_n)_n$, and write $\lim_{n\to\infty} x_n = L$ or $x_n \to L$, if for every (measure of closeness) $\varepsilon > 0$, **eventually** $|x_n - L| < \varepsilon$ (i.e. x_n is within less than ε distance of L).

Rewriting the last definition without using the term **eventually**, we get the following (somewhat dry and hard to comprehend) reformulation:

Definition 2′. We say that $L \in \mathbb{R}$ is a *limit* of a sequence $(x_n)_n$ if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \ge N \ |x_n - L| < \varepsilon.$$

It is also worth noting that the condition $|x_n - L| < \varepsilon$ can be written in various (equivalent) ways, such as:

(i) $-\varepsilon < x_n - L < \varepsilon$
(ii) $-\varepsilon < L - x_n < \varepsilon$
(iii) $L - \varepsilon < x_n < L + \varepsilon$
(iv) $x_n \in (L - \varepsilon, L + \varepsilon)$
(v) $x_n \in B(L, \varepsilon)$, where $B(L, \varepsilon)$ denotes the "open ball around L of radius ε", which simply means $B(L, \varepsilon) := (L - \varepsilon, L + \varepsilon)$.

5. Let $P(n)$ be a mathematical statement for every $n \in \mathbb{N}$. Write down explicitly the negation of the statement "**eventually** $P(n)$ holds".

Solution. $\forall N \in \mathbb{N} \exists n \ge N \neg P(n)$; in words: for every "threshold" $N \in \mathbb{N}$, there is a "bad" index n after that "threshold", at which the property P fails. □

6. Let $n_0 \in \mathbb{N}$. For a sequence $(x_n)_n$, let $(x_n)_{n \ge n_0}$ denote the sequence obtained from $(x_n)_n$ by deleting the first $n_0 - 1$ terms, i.e. $(x_{n_0}, x_{n_0+1}, x_{n_0+2}, ...)$. Prove that $(x_n)_n$ converges to L if and only if $(x_n)_{n \ge n_0}$ converges to L. In other words, the first finitely many terms don't affect the convergence of the sequence.

Solution. This statement is immediately implied by the definition of **eventually** and there is nothing more to write. (I won't demand more on the midterm, don't worry.) $\square$

7. Suppose that $x_n \to L$ and $L > 7$. Prove that **eventually** $x_n > 7$.

Solution. Intuitively, $x_n \to L$ means that no matter how close we want (arbitrary positive distance ε), the members of the sequence $(x_n)_n$ **eventually** get that much close to L. Now how far is L from 7? Their distance is $L - 7$. Thus, we choose our distance $\varepsilon := \frac{L-7}{2}$, so we get that **eventually**

$$L - \varepsilon < x_n < L + \varepsilon.$$

The relevant inequality for us here is the first one because $L - \varepsilon = 7 + \varepsilon$, so $7 < 7 + \varepsilon = L - \varepsilon < x_n$.

`P.S. Drawing a picture always helps.` $\square$

8. For each of the following statements, determine whether they are true or false, and prove your answers.

(a) If a sequence is bounded, it has a limit.

Solution. NOPE, take $(x_n)_n = (0, 1, 0, 1, ...)$. $\square$

(b) The sequence $(0, 1, 0, 1, ...)$ diverges.

Solution. YEP, and to prove it we simply show that no real number L is a limit of this sequence. Fix an arbitrary $L \in \mathbb{R}$. The tricky part here is the realization that we have to consider the following cases separately:

Case 1: $L = 0$. Intuitively, 0 can't be a limit of our sequence because of the 1s. We prove this formally. To show that 0 is not a limit of $(x_n)_n$, we need to find a "bad" $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there is $n \geq N$ with $|x_n| \geq \varepsilon$. In our case, $\varepsilon := 1$ works. Indeed, no matter what N is, taking any even index $n \geq N$ gives $x_n = 1$ so $|x_n| \geq \varepsilon$.

Case 2: $L \neq 0$. Intuitively, a nonzero L can't be a limit of our sequence because of the 0s. We prove this formally. To show that L is not a limit of $(x_n)_n$, we need to find a "bad" $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there is $n \geq N$ with $|x_n - L| \geq \varepsilon$. In our case, we take ε to be the distance between 0 and L, i.e. $\varepsilon := |L - 0| = |L|$. Indeed, no matter what N is, taking any odd index $n \geq N$ gives $x_n = 0$ so $|x_n - L| = |0 - L| = |L| \geq \varepsilon$. $\square$

(c) $\lim_{n\to\infty} \frac{(-1)^n n}{n+1} = -1$.

Solution. NOPE, and showing it is very similar to Case 1 of the previous part. Intuitively, -1 can't be a limit of our sequence because its members at even indices are positive, and hence away from -1 by at least distance 1.

Now formally. To show that -1 is not a limit of $(x_n)_n$, we need to find a "bad" $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there is $n \geq N$ with $|x_n - (-1)| \geq \varepsilon$. In our case, $\varepsilon := 1$ works. Indeed, no matter what N is, taking any even index $n \geq N$ gives $x_n > 0$ so, in particular, $|x_n - (-1)| = |x_n + 1| \geq 1 = \varepsilon$. $\square$

(d) If a sequence is monotone, it has a limit.

Solution. NOPE, take $x_n = n$, then $(x_n)_n$ is unbounded and hence diverges.
This question is design to emphasize the hypothesis of *boundedness* in the Monotone Convergence Theorem. ☐

(e) If $(x_n \cdot y_n)_n$ converges, then at least one of $(x_n)_n$ and $(y_n)_n$ converges.

Solution. NOPE, take $(x_n) = (0, 1, 0, 1, ...)$ and $(y_n) = (1, 0, 1, 0, ...)$, then for all $n \in \mathbb{N}$ $x_n y_n = 0$, so $x_n y_n \to 0$, whereas neither of $(x_n)_n$ and $(y_n)_n$ converges. ☐

(f) If a bounded sequence $(x_n)_n$ is increasing, then it converges to $\sup\{x_n : n \in \mathbb{N}\}$.

Solution. YEP, this is just the statement of the Monotone Convergence Theorem. ☐

9. Do Problems 1, 2(b) and 3 of HW10. If you have time, also do 2(a) and 4.

Solution. Sorry, this is part of the homework. However, here is a hint for 2(b): use 2(a). ☐